MATH2050C Selected Solution to Assignment 1

Section 2.1.

Solution 3.

(a) $2x + 5 = 8$. Subtracting both sides by 5 (or adding -5) to get $2x = 3$ and then divide both side by 2 (or multiply both side by 1/2) to get $x = 3/2$ ($3/2$ is the same as $\frac{3}{2}$).

(b) $x^2 = 2x$. Adding both side by $-2x$ to get $x^2 - 2x = 0$. By (D), $x(x - 2) = 0$. Using $ab = 0$ means a or b equals to 0, we conclude that either $x = 0$ or $x = 2$.

(c) $x^2 - 1 = 3$. Adding -3 to both sides to get $x^2 - 4 = 0$. Then by factorizing $(x+2)(x-2) = 0$. Using $ab = 0$ implies a or b equals to 0, we conclude $x = 2$ or -2.

(d) $(x - 1)(x + 2) = 0$. Immediately get $x = 1$ or -2.

Solution 13. Show that $a^2 + b^2 = 0$ if and only if $a = b = 0$. As $a^2 + b^2 - a^2 = b^2 \geq 0$, we know that $a^2 \geq a^2 + b^2$, so $a^2 \leq 0$. On the other hand, $a^2 \geq 0$. Thus, $a^2 = 0$ which implies $a = 0$. Similarly, $b = 0$. The other direction is obvious.

Solution 16.

(a) $x^2 > 3x + 4$. By factorization this is the same as $(x - 4)(x + 1) > 0$. Therefore, the solution set is $\{x : x > 4 \text{ or } x < -1\}$.

(b) $1 < x^2 < 4$. The solution set for $x^2 > 1$ is $\{x : \ x > 1, \text{ or } x < -1\}$ and the solution set for $x^2 < 4$ is $\{x : \ x \in (-2, 2)\}$. Thus, the solution set for this problem is $(-2, -1) \cup (1, 2)$.

(c) $1/x < x$. When x is positive, this is the same as $1 < x^2$ whose solution set is $\{x : x > 1\}$. When $x < 0$, this is the same as $1 > x^2$ whose solution set is $\{x : x \in (-1, 0)\}$. Hence the solution set for this inequality is $(1, \infty) \cup (-1, 0)$.

(d) $1/x < x^2$. When x is positive, this is the same as $1 < x^3$ whose solution set is $\{x : x > 1\}$. When x is negative, this inequality always holds, so the solution set is $(-\infty, 0)$. Therefore, the solution set for this inequality is $(1, \infty) \cup (-\infty, 0)$.

Solution 23. Show that for positive a, b and $n \in \mathbb{N}$, $a < b$ if and only if $a^n < b^n$.

$\Rightarrow$. Use induction on n. It is obviously true when $n = 1$. Assume that it is true for n. Then $a^{n+1} = aa^n < ab^n$ by induction hypothesis. So, $a^{n+1} < ab^n < b^{n+1}$, done.

$\Leftarrow$. When $a^n < b^n$, by factorization $0 < b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ which implies that $b - a > 0$ since the second factor is always positive.

Supplementary Exercises.

(1). (a) Show that every natural number $n > 1$ can be written uniquely as

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \ ,$$

where p_j's are prime numbers $p_1 < p_2 < \cdots < p_k$ and $n_j \geq 1$. Suggestion: Use induction on n.

(b) Show that for every natural numbers n, m, there exist n', m' with no common factor greater than 1 such that $\dfrac{n}{m} = \dfrac{n'}{m'}$.

Solution. (a). Starting from $n = 2$, a trivial case. Now assuming the factorization is valid for all $k \leq n$, we are going to show that it holds for n. Indeed, if n is a prime number, then $n = n^1$ the factorization holds. If not, let $p > 1$ be one of its prime factor. Then n/p is a number less than n. By induction hypothesis,

$$\frac{n}{p} = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} \ ,$$

so n has a similar decomposition, done. Such factorization is clearly unique (assuming $p_1 < p_2 < \cdots < p_k$).

(b). Easily follows from (a).

Remark. (b) was used in the proof of Theorem 2.1.4.

(2) Denote $\mathbb{Z}_p = \{0, 1, 2, \cdots, p-1\}$ and define addition and multiplication on $\mathbb{Z}_p$ by modulo p, that is, $a + b$ and $a \cdot b$ is equal to the reminder of ordinary $a + b$ and $a \cdot b$ after divided by p respectively. Show that $\mathbb{Z}_p$ satisfies all algebraic properties of the real number system. You may try $p = 5$ first.

Solution. We will work on $p = 5$. It is clear that all conditions (A1)-(A4), (M1)-(M4) and D are satisfied. It suffices to check the existence of multiplicative inverse. In fact, we have $2 \cdot 3 = 1(mod\ 5), 3 \cdot 2 = 1(mod\ 5), 4 \cdot 4 = 1(mod\ 5)$.